The stability of magnetic vortices*

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Abstract

We study the linearized stability of n-vortex ($n \in \mathbf{Z}$) solutions of the magnetic Ginzburg-Landau (or Abelian Higgs) equations. We prove that the fundamental vortices ($n = \pm 1$) are stable for all values of the coupling constant, λ, and we prove that the higher-degree vortices ($|n| \geq 2$) are stable for $\lambda < 1$, and unstable for $\lambda > 1$. This resolves a long-standing conjecture (see, eg, [JT]).

Contents

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1 Introduction

In this paper, we determine the stability of magnetic (or Abelian Higgs) vortices. These are certain critical points of the energy functional

$$E(\psi, A) = \frac{1}{2}\int_{\mathbf{R}^2}\left\{|\nabla_A\psi|^2 + (\nabla\times A)^2 + \frac{\lambda}{4}(|\psi|^2-1)^2\right\} \tag{1}$$

for the fields

$$A : \mathbf{R}^2 \to \mathbf{R}^2 \qquad \text{and} \qquad \psi : \mathbf{R}^2 \to \mathbf{C}.$$

Here $\nabla_A = \nabla - iA$ is the covariant gradient, and $\lambda > 0$ is a coupling constant. For a vector, A, $\nabla \times A$ is the scalar $\partial_1 A_2 - \partial_2 A_1$, and for a scalar ξ, $\nabla\times\xi$ is the vector $(-\partial_2\xi, \partial_1\xi)$. Critical points of $E(\psi, A)$ satisfy the *Ginzburg-Landau* (GL) equations

$$-\Delta_A\psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi = 0 \tag{2}$$

$$\nabla\times\nabla\times A - \Im(\bar{\psi}\nabla_A\psi) = 0 \tag{3}$$

where $\Delta_A = \nabla_A \cdot \nabla_A$.

Physically, the functional $E(\psi, A)$ gives the difference in free energy between the superconducting and normal states near the transition temperature in the Ginzburg-Landau theory. A is the vector potential ($\nabla\times A$ is the induced magnetic field), and ψ is an *order parameter*. The modulus of ψ is interpreted as describing the local density of superconducting Cooper pairs of electrons.

The functional $E(\psi, A)$ also gives the energy of a static configuration in the Yang-Mills-Higgs classical gauge theory on $\mathbf{R}^2$, with abelian gauge group $U(1)$. In this case A is a connection on the principal $U(1)$- bundle $\mathbf{R}^2 \times U(1)$, and ψ is the *Higgs field* (see [JT] for details).

A central feature of the functional $E(\psi, A)$ (and the GL equations) is its infinite-dimensional symmetry group. Specifically, $E(\psi, A)$ is invariant under $U(1)$ *gauge transformations*,

$$\psi \mapsto e^{i\gamma}\psi \tag{4}$$

$$A \mapsto A + \nabla\gamma \tag{5}$$

for any smooth $\gamma : \mathbf{R}^2 \to \mathbf{R}$. In addition, $E(\psi, A)$ is invariant under coordinate translations, and under the coordinate rotation transformation

$$\psi(x) \mapsto \psi(g^{-1}x) \qquad A(x) \mapsto gA(g^{-1}x) \tag{6}$$

for $g \in SO(2)$.

Finite energy field configurations satisfy

$$|\psi| \to 1 \qquad \text{as} \qquad |x| \to \infty \tag{7}$$

which leads to the definition of the *topological degree*, $\deg(\psi)$, of such a configuration:

$$\deg(\psi) = \deg\left(\left. \frac{\psi}{|\psi|} \right|_{|x|=R} : \mathbf{S}^1 \to \mathbf{S}^1 \right)$$

(R sufficiently large). The degree is related to the phenomenon of flux quantization. Indeed, an application of Stokes' theorem shows that a finite-energy configuration satisfies

$$\deg(\psi) = \frac{1}{2\pi} \int_{\mathbf{R}^2} (\nabla \times A).$$

We study, in particular, "radially-symmetric" or "equivariant" fields of the form

$$\psi^{(n)}(x) = f_n(r)e^{in\theta} \qquad A^{(n)}(x) = n\frac{a_n(r)}{r}\hat{x}^{\perp} \tag{8}$$

where (r, θ) are polar coordinates on $\mathbf{R}^2$, $\hat{x}^{\perp} = \frac{1}{r}(-x_2, x_1)^t$, n is an integer, and

$$f_n, a_n : [0, \infty) \to \mathbf{R}.$$

It is easily checked that such configurations (if they satisfy (7)) have degree n. The existence of critical points of this form is well-known (see section 2.1). They are called *n-vortices.*

Our main results concern the stability of these n-vortex solutions. Let

$$L^{(n)} = \text{ Hess } E(\psi^{(n)}, A^{(n)})$$

be the linearized operator for GL around the n-vortex, acting on the space

$$X = L^2(\mathbf{R}^2, \mathbf{C}) \oplus L^2(\mathbf{R}^2, \mathbf{R}^2).$$

The symmetry group of $E(\psi, A)$ gives rise to an infinite-dimensional subspace of $\ker(L^{(n)}) \subset X$ (see section 3.2), which we denote here by Z_{sym}. We say the n-vortex is (linearly) *stable* if for some $c > 0$,

$$L^{(n)}|_{Z_{sym}^{\perp}} \geq c,$$

and *unstable* if $L^{(n)}$ has a negative eigenvalue. The basic result of this paper is the following linearized stability statement:

Theorem 1 *1. (Stability of fundamental vortices)*

For all $\lambda > 0$, the ± 1-vortex is stable.

2. (Stability/instability of higher-degree vortices)

For $|n| \geq 2$, the n-vortex is

$$\begin{cases} \textit{stable} & \textit{for } \lambda < 1 \\ \textit{unstable} & \textit{for } \lambda > 1. \end{cases}$$

Theorem 1 is the basic ingredient in a proof of the nonlinear dynamical stability/instability of the n-vortex for certain dynamical versions of the GL equations. These include the GL gradient flow equations, the Abelian Higgs (Lorentz-invariant) equations, and the Maxwell equations coupled to a nonlinear Schrödinger equation. These dynamical stability results are established in a companion paper ([G2]).

The statement of theorem 1 was conjectured in [JT] on the basis of numerical observations (see [JR]). Bogomolnyi ([B]) gave an argument for instability of vortices for $\lambda > 1$, $|n| \geq 2$. Our result rigorously establishes this property.

The solutions of (2-3) are well-understood in the case of *critical coupling*, $\lambda = 1$. In this case, the *Bogomolnyi method* ([B]) gives a pair of first-order equations whose solutions are global minimizers of $E(\psi, A)$ among fields of fixed degree (and hence solutions of the the GL equations). Taubes ([T1, T2])

has shown that all solutions of GL with $\lambda = 1$ are solutions of these first-order equations, and that for a given degree n, the gauge-inequivalent solutions form a $2|n|$-parameter family. The $2|n|$ parameters describe the locations of the zeros of the scalar field. This is discussed in more detail in [JT] (see also [BGP]) and section 6. We remark that for $\lambda = 1$, an n-vortex solution (8) corresponds to the case when all $|n|$ zeros of the scalar field lie at the origin.

The remainder of this paper is organized as follows. In section 2 we describe in detail various properties of the n-vortex. In particular, we establish an important estimate on the n-vortex profiles which differentiates between the cases $\lambda < 1$ and $\lambda > 1$. In section 3, we introduce the linearized operator, fix the gauge on the space of perturbations, and identify the zero-modes due to symmetry-breaking. Sections 4 through 7 comprise a proof of theorem 1. A block-decomposition for the linearized operator is described in section 4. This approach is similar to that used to study the stability of non-magnetic vortices in [OS1] and [G1]. In section 5, we establish the positivity of certain blocks (those corresponding to the radially-symmetric variational problem, and those containing the translational zero-modes) for all λ, which completes the stability proof for the ± 1-vortices. The basic techniques are the characterization of symmetry-breaking in terms of zero-modes of the Hessian (or linearized operator), and a Perron-Frobenius type argument, based on a version of the maximum principle for systems (proposition 6), which shows that the translational zero-modes correspond to the bottom of the spectrum of the linearized operator. A more careful analysis is needed for $|n| \geq 2$. This requires us to review some aspects of the critical case ($\lambda = 1$) in section 6. The stability/instability proof for $|n| \geq 2$ is completed in section 7. We use an extension of Bogomolnyi's instability argument, and another application of the Perron-Frobenius theory.

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2 The n-vortex

In this section we discuss the existence, and properties, of n-vortex solutions.

2.1 Vortex solutions

The existence of solutions of (GL) of the form (8) is well-known:

Theorem 2 (Vortex Existence; [P, BC]) *For every integer n, there is a solution*

$$\psi^{(n)}(x) = f_n(r)e^{in\theta} \qquad A^{(n)}(x) = n\frac{a_n(r)}{r}\hat{x}^{\perp} \tag{9}$$

of the variational equations (2)-(3). In particular, the radial functions (f_n, a_n) minimize the radial energy functional

$$E_r^{(n)}(f,a) = \frac{1}{2}\int_0^\infty \left\{ (f')^2 + n^2\frac{(1-a)^2 f^2}{r^2} + n^2\frac{(a')^2}{r^2} + \frac{\lambda}{4}(f^2-1)^2 \right\} rdr \tag{10}$$

(which is the full energy functional (1) restricted to fields of the form (8)) in the class

$$\{f, a : [0,\infty) \to \mathbf{R} \mid 1 - f \in H_1(rdr), \frac{a}{r} \in L^2_{loc}(rdr), \frac{a'}{r} \in L^2(rdr)\}.$$

The functions f_n, a_n are smooth, and have the following properties (for $n \neq 0$):

1. $0 < f_n < 1$, $0 < a_n < 1$ *on* $(0,\infty)$
2. $f_n', a_n' > 0$
3. $f_n \sim cr^n$, $a_n \sim dr^2$, *as* $r \to 0$ *($c > 0$ and $d > 0$ are constants)*
4. $1 - f_n$, $1 - a_n \to 0$ *as* $r \to \infty$, *with an exponential rate of decay.*

We call ($\psi^{(n)}$, $A^{(n)}$) an *n-vortex* (centred at the origin).

It follows immediately that the functions f_n and a_n satisfy the ODEs

$$-\Delta_r f_n + \frac{n^2(1-a_n)^2}{r^2}f_n + \frac{\lambda}{2}(f_n^2-1)f_n = 0 \tag{11}$$

and

$$-a_n'' + \frac{a_n'}{r} - f_n^2(1-a_n) = 0. \tag{12}$$

Remark 1 *To our knowledge, it is not known if solutions of the form (8) are unique. In the appendix, we show that for* $\lambda \geq 2n^2$*, any such solution minimizes* $E_r^{(n)}$.

Remark 2 *The functions* f_n *and* a_n *also depend on* λ*, but we suppress this dependence for ease of notation. When it will cause no confusion, we will also drop the subscript* n.

Remark 3 *The discrete symmetry* $\psi \mapsto \bar{\psi}$, $A \mapsto -A$ *of (GL) interchanges* $(\psi^{(n)}, A^{(n)})$ *and* $(\psi^{(-n)}, A^{(-n)})$*. Thus, we can assume* $n \geq 0$.

2.2 An estimate on the vortex profiles

The following inequality, relating the exponentially decaying quantities f' and $1-a$, plays a crucial role in the stability/instability proof.

Proposition 1 *We have*

$$\begin{cases} f'(r) > \frac{n(1-a(r))}{r} f(r) & \text{for} \quad \lambda < 1 \\ f'(r) < \frac{n(1-a(r))}{r} f(r) & \text{for} \quad \lambda > 1 \end{cases} \tag{13}$$

Proof: Define $e(r) \equiv f'(r) - \frac{n(1-a(r))}{r} f(r)$. The properties listed in theorem 2 imply that $e(r) \to 0$ as $r \to 0$ and as $r \to \infty$. Using the ODEs ((11)-(12)) we can derive the equation

$$(-\Delta_r + \alpha)e + \frac{e}{f}e' = (1-\lambda)f^2 f'$$

where

$$\alpha(r) = \frac{1+n(1-a)}{r^2}(1 + \frac{rf'}{f}) + f^2 + \frac{na'}{r} > 0$$

and the result follows from the maximum principle. □

3 The linearized operator

In this section, we introduce the linearized operator (or Hessian) around the n-vortex, and identify its symmetry zero-modes.

3.1 Definition of the linearized operator

We work on the real Hilbert space

$$X = L^2(\mathbf{R}^2; \mathbf{C}) \oplus L^2(\mathbf{R}^2; \mathbf{R}^2)$$

with inner-product

$$< (\xi, B), (\eta, C) >_X = \int_{\mathbf{R}^2} \{\Re(\bar{\xi}\eta) + B \cdot C\}.$$

We define the linearized operator, $L_{\psi,A}$ (= the Hessian of $E(\psi, A)$) at a solution (ψ, A) of (2)-(3) through the quadratic form

$$\frac{\partial^2}{\partial\epsilon\partial\delta} E(\psi + \epsilon\xi + \delta\eta, A + \epsilon B + \delta C)|_{\epsilon=\delta=0} = \langle(\eta, C) L_{\psi,A}(\xi, B)\rangle_X$$

for all (ξ, B), (η, C), $\in X$. The result is

$$L_{\psi,A} \begin{pmatrix} \xi \\ B \end{pmatrix} = \begin{pmatrix} [-\Delta_A + \frac{\lambda}{2}(2|\psi|^2 - 1)]\xi + \frac{\lambda}{2}\psi^2\bar{\xi} + i[2\nabla_A\psi + \psi\nabla] \cdot B \\ \Im([\bar{\nabla_A}\psi - \bar{\psi}\nabla_A]\xi) + (-\Delta + \nabla\nabla + |\psi|^2) \cdot B \end{pmatrix}.$$

3.2 Symmetry zero-modes

We identify the part of the kernel of the operator

$$L^{(n)} \equiv L_{\psi^{(n)},A^{(n)}}$$

which is due to the symmetry group.

Proposition 2 *We have*

1.

$$L^{(n)} \begin{pmatrix} i\gamma\psi^{(n)} \\ \nabla\gamma \end{pmatrix} = 0 \tag{14}$$

for any $\gamma : \mathbf{R}^2 \to \mathbf{R}$

2.

$$L^{(n)} \begin{pmatrix} \partial_j\psi^{(n)} \\ \partial_j A^{(n)} \end{pmatrix} = 0 \tag{15}$$

for $j = 1, 2.$

Proof: We use the basic result that the generator of a one-parameter group of symmetries of $E(\psi, A)$, applied to the n-vortex, lies in the kernel of $L^{(n)}$. The vector in (14) is easily seen to be the generator of a one-parameter family of gauge transformations (4-5) applied to the n-vortex. Similarly, the vector in (15) is the generator of coordinate translations applied to the n-vortex. $\square$

Remark 4 *Applying the generator of the coordinate rotational symmetry (6) to the n-vortex gives us nothing new, it is contained in the gauge-symmetry case.*

We define Z_{sym} to be the subspace of X spanned by the L^2 zero-modes described in proposition 2. We recall that the n-vortex is called *stable* if there is a constant $c > 0$ such that

$$L^{(n)}|_{Z_{sym}^{\perp}} \geq c, \tag{16}$$

and *unstable* if $L^{(n)}$ has a negative eigenvalue.

3.3 Gauge fixing

In order to remove the infinite dimensional kernel of $L^{(n)}$ arising from gauge symmetry, we restrict the class of perturbations. Specifically, we restrict $L^{(n)}$ to the space of those perturbations $(\xi, B) \in X$ which are orthogonal to the L^2 gauge zero-modes (14). That is,

$$\left\langle \begin{pmatrix} i\gamma\psi^{(n)} \\ \nabla\gamma \end{pmatrix}, \begin{pmatrix} \xi \\ B \end{pmatrix} \right\rangle_X = 0$$

for all γ. Integration by parts gives the gauge condition

$$\Im(\overline{\psi^{(n)}}\xi) = \nabla \cdot B. \tag{17}$$

As is done in [S], we consider a modified quadratic form $\tilde{L}^{(n)}$, defined by

$$< \alpha, \tilde{L}^{(n)}\alpha > = < \alpha, L^{(n)}\alpha > + \int (\Im(\overline{\psi^{(n)}}\xi) - \nabla \cdot B)^2$$

for $\alpha = (\xi, B) \in X$. Clearly, $\tilde{L}^{(n)}$ agrees with $L^{(n)}$ on the subspace of X specified by the gauge condition (17). This modification has the important effect of shifting the essential spectrum away from zero (see (26)). A straightforward computation gives the following expression for $\tilde{L}^{(n)}$:

$$\tilde{L}^{(n)} \begin{pmatrix} \xi \\ B \end{pmatrix} = \begin{pmatrix} [-\Delta_A + \frac{\lambda}{2}(2|\psi|^2 - 1) + \frac{1}{2}|\psi|^2]\xi + \frac{1}{2}(\lambda - 1)\psi^2\bar{\xi} + 2i\nabla_A\psi \cdot B \\ 2\Im[\bar{\nabla_A}\psi\xi] + [-\Delta + |\psi|^2]B \end{pmatrix}.$$

To establish theorem 1, it suffices to prove that $\tilde{L}^{(n)} \geq c > 0$ on the subspace of X orthogonal to the translational zero-modes (15).

$\tilde{L}^{(n)}$ is a real-linear operator on X. It is convenient to identify $L^2(\mathbf{R}^2;\mathbf{R}^2)$ with $L^2(\mathbf{R}^2;\mathbf{C})$ through the correspondence

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \leftrightarrow B^c \equiv B_1 - iB_2, \tag{18}$$

and then to complexify the space $X \mapsto \tilde{X} = [L^2(\mathbf{R}^2;\mathbf{C})]^4$ via

$$(\xi, B) \mapsto (\xi, \bar{\xi}, B^c, \bar{B}^c). \tag{19}$$

As a result, $\tilde{L}^{(n)}$ is replaced by the complex-linear operator

$$\tilde{\tilde{L}}^{(n)} = \text{ diag } \{-\Delta_A, -\overline{\Delta_A}, -\Delta, -\Delta\} + V^{(n)}$$

where

$$V^{(n)} = \begin{pmatrix} \frac{\lambda}{2}(2|\psi|^2 - 1) + \frac{1}{2}|\psi|^2 & \frac{1}{2}(\lambda - 1)\psi^2 & -i(\partial_A^*\psi) & i(\partial_A\psi) \\ \frac{1}{2}(\lambda - 1)\bar{\psi}^2 & \frac{\lambda}{2}(2|\psi|^2 - 1) + \frac{1}{2}|\psi|^2 & -i(\bar{\partial_A}\psi) & i(\bar{\partial_A^*}\psi) \\ i(\bar{\partial_A^*}\psi) & i(\partial_A\psi) & |\psi|^2 & 0 \\ -i(\bar{\partial_A}\psi) & -i(\partial_A^*\psi) & 0 & |\psi|^2 \end{pmatrix}.$$

Here we have used the notation

$$\partial_A \equiv \partial_z - iA$$

where $\partial_z = \partial_1 - i\partial_2$ (and the superscript c has been dropped from the complex function A obtained from the vector-field A via (18)).

The components of $V^{(n)}$ are bounded, and it follows from standard results ([RSII]) that $\tilde{\tilde{L}}^{(n)}$ is a self-adjoint operator on $\tilde{X}$, with domain

$$D(\tilde{\tilde{L}}^{(n)}) = [H_2(\mathbf{R}^2;\mathbf{C})]^4$$

4 Block decomposition

We write functions on $\mathbf{R}^2$ in polar coordinates. Precisely,

$$\tilde{X} = [L^2(\mathbf{R}^2;\mathbf{C})]^4 = [L^2_{rad} \otimes L^2(\mathbf{S}^1;\mathbf{C})]^4 \tag{20}$$

where $L^2_{rad} \equiv L^2(\mathbf{R}^+, rdr)$.

Let $\rho_n : U(1) \to Aut([L^2(\mathbf{S}^1;\mathbf{C})]^4)$ be the representation whose action is given by

$$\rho_n(e^{i\theta})(\xi, \eta, B, C)(x) = (e^{in\theta}\xi, e^{-in\theta}\eta, e^{-i\theta}B, e^{i\theta}C)(R_{-\theta}x)$$

where R_α is a counter-clockwise rotation in $\mathbf{R}^2$ through the angle α. It is easily checked that the linearized operator $\tilde{\tilde{L}}^{(n)}$ commutes with $\rho_n(g)$ for any $g \in U(1)$. It follows that $\tilde{\tilde{L}}^{(n)}$ leaves invariant the eigenspaces of $d\rho_n(s)$ for any $s \in i\mathbf{R} = Lie(U(1))$. The resulting block decomposition of $\tilde{\tilde{L}}^{(n)}$, which is described in this section, is essential to our analysis. In particular, the translational zero-modes each lie within a single subspace of this decomposition.

4.1 The decomposition of $L^{(n)}$

In what follows, we define, for convenience, $b(r) = \frac{n(1-a(r))}{r}$.

Proposition 3 *There is an orthogonal decomposition*

$$\tilde{X} = \bigoplus_{m \in \mathbf{Z}} (e^{i(m+n)\theta} L^2_{rad} \oplus e^{i(m-n)\theta} L^2_{rad} \oplus -ie^{i(m-1)\theta} L^2_{rad} \oplus ie^{i(m+1)\theta} L^2_{rad}), \tag{21}$$

under which the linearized operator around the vortex, $\tilde{\tilde{L}}^{(n)}$, *decomposes as*

$$\tilde{\tilde{L}}^{(n)} = \bigoplus_{m \in \mathbf{Z}} \hat{L}^{(n)}_m$$

where

$$\hat{L}^{(n)}_m = -\Delta_r(Id) + \hat{V}^{(n)}_m \tag{22}$$

with

$$\hat{V}^{(n)}_m = \frac{1}{r^2} \; diag \; \{[m + n(1-a)]^2, [m - n(1-a)]^2, [m-1]^2, [m+1]^2\} + V'$$

and

$$V' = \begin{pmatrix} \frac{\lambda}{2}(2f^2 - 1) + \frac{1}{2}f^2 & \frac{1}{2}(\lambda - 1)f^2 & f' - bf & -[f' + bf] \\ \frac{1}{2}(\lambda - 1)f^2 & \frac{\lambda}{2}(2f^2 - 1) + \frac{1}{2}f^2 & -[f' + bf] & f' - bf \\ f' - bf & -[f' + bf] & f^2 & 0 \\ -[f' + bf] & f' - bf & 0 & f^2 \end{pmatrix}.$$

Proof: The decomposition (21) of $\tilde{X}$ follows from the usual Fourier decomposition of $L^2(\mathbf{S}^1; \mathbf{C})$, and the relation (20). An easy computation shows that $\tilde{\tilde{L}}^{(n)}$ preserves the space of vectors of the form

$$(\xi e^{i(m+n)\theta}, \eta e^{i(m-n)\theta}, -i\alpha e^{i(m-1)\theta}, i\beta e^{i(m+1)\theta}) \tag{23}$$

and that it acts on such vectors via (22). $\square$

It follows that $\hat{L}^{(n)}_m$ is self-adjoint on $[L^2_{rad}]^4$. It will also be convenient to work with a rotated version of the operator $\hat{L}^{(n)}_m$,

$$L^{(n)}_m \equiv \begin{cases} R\hat{L}^{(n)}_m R^T & m \geq 0 \\ R'\hat{L}^{(n)}_m (R')^T & m < 0 \end{cases}$$

where

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \qquad R' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

We have

$$L_m^{(n)} = -\Delta_r(Id) + V_m^{(n)} \tag{24}$$

where

$$V_m^{(n)} = \begin{pmatrix} \frac{m^2}{r^2} + b^2 + \frac{\lambda}{2}(3f^2-1) & -2|m|\frac{b}{r} & -2bf & 0 \\ -2|m|\frac{b}{r} & \frac{m^2}{r^2} + b^2 + \frac{\lambda}{2}(f^2-1) + f^2 & 0 & -2f' \\ -2bf & 0 & \frac{m^2+1}{r^2} + f^2 & -2\frac{|m|}{r^2} \\ 0 & -2f' & -2\frac{|m|}{r^2} & \frac{m^2+1}{r^2} + f^2 \end{pmatrix}.$$

4.2 Properties of $L_m^{(n)}$

Proposition 4 *We have the following:*

1.
$$L_m^{(n)} = L_{-m}^{(n)} \tag{25}$$

2.
$$\sigma_{ess}(L_m^{(n)}) = [\min(1,\lambda), \infty) \tag{26}$$

3. *For* $|n| = 1$ *and* $m \geq 2$,
$$L_m^{(n)} - L_1^{(n)} \geq 0 \tag{27}$$

   with no zero-eigenvalue.

Proof: The first statement is obvious. The second statement follows in a standard way from the fact that

$$\lim_{r\to\infty} V_m^{(n)}(r) = \text{ diag } \{\lambda, 1, 1, 1\}$$

To prove the third statement, we compute

$$\hat{L}_m^{(n)} - \hat{L}_1^{(n)} = \frac{m-1}{r^2}\ \mathrm{diag}\ \{m + 2n(1-a),\ m - 2n(1-a),\ m-1,\ m+3\}$$

which is non-negative, with no zero-eigenvalue for $m \geq 2$, $n = 1$. $\Box$

Remark 5 *In light of (25), we can assume from now on that $m \geq 0$. This degeneracy is a result of the complexification (19) of the space of perturbations.*

4.3 Translational zero-modes

The gauge fixing (section 3.3) has eliminated the zero-modes arising from gauge symmetry. The translational zero-modes remain.

As written in (15), the translational zero-modes fail to satisfy the gauge condition (17). Further, they do not lie in L^2. A straightforward computation shows that if we adjust the vectors in (15) by gauge zero-modes given by (14) with $\gamma = -A_j$, $j = 1, 2$, we obtain

$$T_1 = \begin{pmatrix} (\nabla_A \psi)_1 \\ (\nabla \times A) e_2 \end{pmatrix} \qquad T_2 = \begin{pmatrix} (\nabla_A \psi)_2 \\ -(\nabla \times A) e_1 \end{pmatrix}$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. T_1 and T_2 satisfy (17), and are zero-modes of the linearized operator. Note also that $T_{\pm 1}$ decay exponentially as $|x| \to \infty$, and hence lie in L^2.

It is easily checked that $T_1 \pm iT_2$ lie in the $m = \pm 1$ blocks for $\hat{L}_m^{(n)}$. After rotation by R, we have

$$L_{\pm 1}^{(n)} T = 0$$

where

$$T = (f', bf, n\frac{a'}{r}, n\frac{a'}{r}).$$

5 Stability of the fundamental vortices

In this section we prove the first part of theorem 1. Specifically, we show that for some $c > 0$, $L_m^{(\pm 1)} \geq c$ for $m \neq 1$, and $L_1^{(\pm 1)}|_{T^\perp} \geq c$. In light of the discussions in sections 3.3, 4.1, and 4.3, this will establish the stability of the ± 1-vortices.

5.1 Non-negativity of $L_0^{(n)}$ and radial minimization

Proposition 5 *$L_0^{(n)} \geq 0$ for all λ.*

Proof:

From the expression (24) we see that $L_0^{(n)}$ breaks up:

$$L_0^{(n)} = N_0 \oplus M_0 \tag{28}$$

(abusing notation slightly) where

$$M_0 = -\Delta_r(Id) + W_0$$

with

$$W_0 = \begin{pmatrix} b^2 + \frac{\lambda}{2}(3f_n^2 - 1) & -2bf \\ -2bf & \frac{1}{r^2} + f^2 \end{pmatrix}$$

and

$$N_0 = \begin{pmatrix} -\Delta_r + b^2 + \frac{\lambda}{2}(f^2 - 1) + f^2 & -2f' \\ -2f' & -\Delta_r + \frac{1}{r^2} + f^2. \end{pmatrix}$$

An easy computation shows that M_0 is precisely the Hessian of the radial energy, $HessE_r^{(n)}$ (see (10)). Since the n-vortex minimizes $E_r^{(n)}$, we have $M_0 \geq 0$. It remains to show $N_0 \geq 0$. We establish the stronger result, $N_0 > 0$. Note that

$$N_0 = G_0^* G_0$$

where

$$G_0 = \begin{pmatrix} \partial_r - f'/f & f \\ f & \partial_r + 1/r \end{pmatrix}$$

In fact, G_0 has no zero-eigenvalue. To see this, we first remark that G_0 is a relatively compact perturbation of $G_0|_{\lambda=1}$, due to the exponential decay of the field components. It follows from an index-theoretic calculation done in [W, S], that $G_0|_{\lambda=1}$ is Fredholm, with index 0. We conclude that the same is true of G_0 (for any λ). Finally, it is a simple matter to check that G_0^* has trivial kernel. If

$$G_0^* \begin{pmatrix} \xi \\ \beta \end{pmatrix} = 0$$

it follows that

$$(-\Delta_r + f^2)\beta = 0$$

and hence that $\beta = 0$, and so $\xi = 0$. The relation $N_0 > 0$ follows from this, and the the fact that $\sigma_{ess}(N_0) = [1, \infty)$. $\square$

5.2 A maximum principle argument

Removing the equality in proposition 5 requires more work. First, we establish an extension of the maximum principle to systems (see, eg, [LM, PA] for related results). We will use this also in the proof that the the translational zero-mode is the ground state of $L_1^{(n)}$ (section 5.4).

Proposition 6 *Let L be a self-adjoint operator on $L^2(\mathbf{R}^n; \mathbf{R}^d)$ of the form*

$$L = -\Delta(Id) + V$$

where V is a $d \times d$ matrix-multiplication operator with smooth entries. Suppose that $L \geq 0$ and that for $i \neq j$, $V_{ij}(x) \leq 0$ for all x. Further, suppose V is irreducible in the sense that for any splitting of the set $\{1, \dots, d\}$ into disjoint sets S_1 and S_2, there is an $i \in S_1$ and a $j \in S_2$ with $V_{ij}(x) < 0$ for all x. Finally, suppose that $L\xi = \eta \in L^2$ with $\eta \geq 0$ component-wise, and $\xi \not\equiv 0$. Then either

1. $\xi > 0$ *or*

2. $\eta \equiv 0$ *and* $\xi < 0$.

Proof: We write $\xi = \xi^+ - \xi^-$ with $\xi^+, \xi^- \geq 0$ component-wise, and compute

$$0 \;\leq\; <\xi^-, L\xi^-> \;=\; <\xi^-, L\xi^+> - <\xi^-, L\xi> .$$

Since ξ_j^+ and ξ_j^- have disjoint support, we have

$$r.h.s = \sum_{j \neq k} <\xi_j^-, V_{jk}\xi_k^+> - <\xi^-, \eta> \;\leq 0.$$

Thus we have

1. $0 = \; <\xi^-, L\xi^->$
2. $0 = \; <\xi_j^-, V_{jk}\xi_k^+>$ for all $j \neq k$

Since $L \geq 0$, the first of these implies $L\xi^- = 0$ and hence $L\xi^+ = \eta$. So if $\eta \not\equiv 0$, then $\xi^+ \not\equiv 0$. If $\eta \equiv 0$ and $\xi^+ \equiv 0$, replace ξ with $-\xi$ in what follows. An application of the strong maximum principle (eg. [GT], Thm. 8.19) to each component of the equation

$$L\xi^+ = \eta$$

now allows us to conclude that for each k, either $\xi_k^+ > 0$ or $\xi_k^+ \equiv 0$. We know that for some k, $\xi_k^+ > 0$. Looking back at the second listed equation above, and using the irreducibility of V, we then see that $\xi_j^- \equiv 0$ for all j. Finally, we can easily rule out the possibility $\xi_k \equiv 0$ for some k, by looking back at the equation satisfied by ξ_k. Thus we have $\xi > 0$. $\square$

5.3 Positivity of $L_0^{(n)}$

Now we apply proposition 6 to show $M_0 > 0$. The trick here is to find a function ξ which satisfies $M_0\xi \geq 0$. This allows us to rule out the existence of a zero-eigenvector, which would be positive

by proposition 6. To obtain such a ξ, we differentiate the vortex with respect to the parameter λ. Specifically, differentiation of the Ginzburg-Landau equations with respect to λ results in

$$M_0\xi = \eta \tag{29}$$

where

$$\xi = \begin{pmatrix} \partial_\lambda f \\ n\partial_\lambda a/r \end{pmatrix}$$

and

$$\eta = \begin{pmatrix} \frac{1}{2}(1-f^2)f \\ 0 \end{pmatrix} \geq 0.$$

We can now establish

Proposition 7 *For all* λ*,* $L_0^{(n)} \geq c > 0$*.*

Proof: We have already shown in the proof of proposition 5, that $N_0 > 0$ and $M_0 \geq 0$. Hence, due to (28) and (26), it suffices to show that $Null(M_0) = \{0\}$. Suppose $M_0\zeta = 0$, $\zeta \not\equiv 0$. Proposition 6 then implies $\zeta > 0$ (or else take $-\zeta$). Now

$$0 = \; < M_0\zeta, \xi > \; = \; < \zeta, M_0\xi > \; = \; < \zeta, \eta > \; > \; 0$$

gives a contradiction. □

Remark 6 *Proposition 6 applied to equation (29) also gives* $\xi > 0$*. That is, the vortex profiles increase monotonically with* λ*. This can be used to show that the rescaled vortex* $(f_n(r/\sqrt{\lambda}), a_n(r/\sqrt{\lambda}))$ *converges as* $\lambda \to \infty$ *to* $(f^*, 0)$*, where* f^* *is the (profile of) the* n*-vortex solution of the ordinary GL equation:* $-\Delta_r f^* + n^2 f^*/r^2 + (f^{*2}-1)f^* = 0$*. This result was established by different means in [ABG].*

5.4 Positivity of $L_1^{(\pm 1)}$

Proposition 8 $L_1^{(\pm 1)} \geq 0$ *with non-degenerate zero-eigenvalue given by* T*.*

Proof: Let $\mu = inf\,spec L_1^{(\pm 1)} \leq 0$, which is an eigenvalue by (26). Suppose $L_1^{(\pm 1)} S = \mu S$. Applying proposition 6 to $L_1^{(\pm 1)} - \mu$ (note that V_1^1 satisfies the irreducibility requirement) gives $S > 0$ (or $S < 0$). Further, μ is non-degenerate, as if μ were degenerate, we would have two strictly positive eigenfunctions which are orthogonal, an impossibility. Now if $\mu < 0$, we have $< S, T >= 0$, which is also impossible. Thus S is a multiple of T, and $\mu = 0$. $\square$

5.5 Completion of stability proof for $n = \pm 1$

We are now in a position to complete the proof of the first statement of theorem 1. By proposition 7, $L_0^{(\pm 1)} \geq c > 0$. By proposition 8 and (26), $L_1^{(\pm 1)}|_{T^\perp} \geq \tilde{c} > 0$. Finally, by (27), $L_m^{(\pm 1)} \geq c' > 0$ for $|m| \geq 2$. It follows from proposition 3 that $\tilde{L}^{(n)} \geq c > 0$ on the subspace of X orthogonal to the translational zero-modes. By the discussion of section 3.3, this gives theorem 1 for $n = \pm 1$. $\square$

6 The critical case, $\lambda = 1$

In order to prove the remainder of theorem 1, we exploit some results from the $\lambda = 1$ case.

6.1 The first-order equations

Following [B], we use an integration by parts to rewrite the energy (1) as

$$E(\psi, A) = \frac{1}{2} \int_{\mathbf{R}^2} \{ |\partial_A \psi|^2 + [\nabla \times A + \frac{1}{2}(|\psi|^2 - 1)]^2 + \frac{1}{4}(\lambda - 1)(|\psi|^2 - 1)^2 \} + \pi \mathrm{deg}(\psi) \qquad (30)$$

(recall, since we work in dimension two, $\nabla \times A$ is a scalar) where $\mathrm{deg}(\psi)$ is the topological degree of ψ, defined in the introduction. We assume, without loss of generality, that $deg(\psi) \geq 0$. Clearly, when $\lambda = 1$, a solution of the first-order equations

$$\partial_A \psi = 0 \qquad (31)$$

$$\nabla \times A + \frac{1}{2}(|\psi|^2 - 1) = 0 \qquad (32)$$

minimizes the energy within a fixed topological sector, $\deg(\psi) = n$, and hence is stable. Note that we have identified the vector-field A with a complex field as in (18).

The n-vortices (9) are solutions of these equations (when $\lambda = 1$). Specifically,

$$n\frac{a'}{r} = \frac{1}{2}(1 - f^2) \tag{33}$$

and

$$f' = n\frac{(1-a)f}{r}. \tag{34}$$

In fact, it is shown in [T2] that for $\lambda = 1$, any solution of the variational equations solves the first-order equations (31-32).

Beginning from expression (30) for the energy, the variational equations (previously written as (2-3)) can be written as

$$\partial_A^*[\partial_A\psi] + \psi[\nabla \times A + \frac{1}{2}(|\psi|^2 - 1)] + \frac{1}{2}(\lambda - 1)(|\psi|^2 - 1)\psi = 0 \tag{35}$$

$$i\bar{\psi}[\partial_A\psi] - i\partial_{\bar{z}}[\nabla \times A + \frac{1}{2}(|\psi|^2 - 1)] = 0 \tag{36}$$

(here $\partial_A^* \equiv -\partial_z + iA$ is the adjoint of ∂_A).

6.2 First-order linearized operator

We show that the linearized operator at $\lambda = 1$ is the square of the linearized operator for the first-order equations.

Linearizing the first-order equations (31-32) about a solution, (ψ, A) (of the first-order equations) results in the following equations for the perturbation, $\alpha \equiv (\xi, B)$:

$$\partial_A\xi - iB\psi = 0$$

$$\nabla \times B + \Re(\bar{\psi}\xi) = 0.$$

Now using $-i\partial_z B = \nabla \times B - i(\nabla \cdot B)$, and adding in the gauge condition (17), we can rewrite this as

$$L_1 \alpha = 0 \tag{37}$$

where

$$L_1 = \begin{pmatrix} \partial_A & -i\psi \\ \bar{\psi} & -i\partial_z \end{pmatrix}.$$

If we linearize the full (second order) variational equations (in the form (35-36)) around (ψ, A), we obtain

$$\partial_A^*[\partial_A \xi - iB\psi] + i\bar{B}[\partial_A \psi] + \psi[\nabla \times B + \Re(\bar{\psi}\xi)]$$
$$+\xi[\nabla \times A + \frac{1}{2}(|\psi|^2 - 1)] + \frac{1}{2}(\lambda - 1)[(|\psi|^2 - 1)\xi + 2\psi\Re(\bar{\psi}\xi)] = 0$$

and

$$i\bar{\psi}[\partial_A \xi - iB\psi] + i\bar{\xi}[\partial_A \psi] - i\partial_{\bar{z}}[\nabla \times B + \Re(\bar{\psi}\xi)] = 0.$$

Proposition 9 *When $\lambda = 1$, these linearized equations can also be written*

$$L_1^* L_1 \alpha = 0$$

Proof: This is a simple computation using the fact that the first-order equations (31-32) hold. □

This relation holds also on the level of the blocks. A straightforward computation gives

$$L_m^{(n)}|_{\lambda=1} = F_m^* F_m$$

where

$$F_m = \begin{pmatrix} \partial_r - b & \frac{m}{r} & 0 & f \\ \frac{m}{r} & \partial_r - b & -f & 0 \\ 0 & -f & \partial_r + 1/r & \frac{m}{r} \\ f & 0 & \frac{m}{r} & \partial_r + 1/r \end{pmatrix}$$

6.3 Zero-modes for $\lambda = 1$

It was predicted in [W] (and proved rigorously in [S]) that for $\lambda = 1$, the linearized operator around any degree-n solution of the first-order equations has a $2|n|$-dimensional kernel (modulo gauge transformations). This kernel arises because the Taubes solutions form a $2|n|$-parameter family, and all have the same energy. The zero-eigenvalues are identified in [B], and we describe them here. Let χ_m be the unique solution of

$$(-\Delta_r + \frac{m^2}{r^2} + f^2)\chi_m = 0$$

on $(0, \infty)$ with

$$\chi_m \sim r^{-m} \quad \text{as} \quad r \to 0$$

and

$$\chi_m \to 0 \quad \text{as} \quad r \to \infty$$

for $m = 1, 2, \ldots, n$. Then it is easy to check that

$$F_{\pm m} W_m = 0 \tag{38}$$

where

$$W_m = \begin{pmatrix} f\chi_m \\ f\chi_m \\ -(\chi_m' + m\chi_m/r) \\ -(\chi_m' + m\chi_m/r) \end{pmatrix}.$$

We remark that

$$\chi_1 = \frac{1-a}{r}$$

and it is easily verified that for $\lambda = 1$, $W_{\pm 1} = T$ are the translational zero-modes.

7 The (in)stability proof for $|n| \geq 2$

Here we complete the proof of theorem 1.

The idea is to decompose $L_m^{(n)}$ into a sum of two terms, each of which has the same (translational) zero-mode (for $m = 1$) as $L_m^{(n)}$. One term is manifestly positive, and the other satisfies restrictions of Perron-Frobenius theory.

We begin by modifying F_m, and defining, for any λ,

$$\tilde{F}_m \equiv \begin{pmatrix} (\partial_r - \frac{f'}{f}) \cdot q & \frac{m}{r} & 0 & f \\ \frac{m}{r} q & \partial_r - \frac{f'}{f} & -f & 0 \\ 0 & -f & \partial_r + 1/r & \frac{m}{r} \\ fq & 0 & -\frac{m}{r} & \partial_r + 1/r \end{pmatrix}$$

where we have defined

$$q(r) \equiv \frac{n(1-a)f}{rf'} \tag{39}$$

and $\partial_r \cdot q$ denotes an operator composition. By (34), we have $q \equiv 1$ for $\lambda = 1$. We also set, for $m = 1, \dots, n$,

$$\tilde{W}_m = \begin{pmatrix} q^{-1} f \chi_m \\ f \chi_m \\ -(\chi_m' + m\frac{\chi_m}{r}) \\ -(\chi_m' + m\frac{\chi_m}{r}) \end{pmatrix}$$

Now $\tilde{W}_m$ has the following properties:

1. $\tilde{W}_{\pm 1}$ is the translational zero-mode T for all λ
2. when $\lambda = 1$, $\tilde{W}_m = W_m$, $m = \pm 1, \dots, \pm n$, give the $2n$ zero-modes (38) of the linearized operator.

These W_m were chosen in [B] as candidates for directions of energy decrease (for $|m| \geq 2$) when $\lambda > 1$. Intuitively, we think of $\tilde{W}_m$ as a perturbation that tends to break the n-vortex into separate vortices of lower degree.

Now, $\tilde{F}_m$ was designed to have the following properties:

1. $\tilde{F}_m = F_m$ when $\lambda = 1$ (this is clear)

2. $\tilde{F}_m \tilde{W}_m = 0$ for all m and λ (this is easily checked).

A straightforward computation gives

$$L_m^{(n)} = \tilde{F}_m^* \tilde{F}_m + J M_m \tag{40}$$

where $J = \text{diag}\{1,0,0,0\}$ and

$$M_m = l_m - q l_m q + (\lambda - q^2) f^2$$

with

$$l_m = -\Delta_r + \frac{m^2}{r^2} + b^2 + \frac{\lambda}{2}(f^2 - 1).$$

By construction, when $m = 1$, the second term in the decomposition (40) must have a zero-mode corresponding to the original translational zero-mode. In fact, one can easily check that $M_1 f' = 0$.

Proposition 10 *For $|n| \geq 2$, M_1 has a non-degenerate zero-eigenvalue corresponding to f', and*

$$\begin{cases} M_1 \geq 0 & \lambda < 1 \\ M_1 \leq 0 & \lambda > 1 \end{cases}$$

on L^2_{rad}.

Proof: We recall inequality (13), which implies that for $\lambda < 1$, $q < 1$, and for $\lambda > 1$, $q > 1$. The operator M_1 is of the form

$$M_1 = (1 - q^2)(-\Delta_r) + \text{ first order } + \text{ multiplication.} \tag{41}$$

One can show that M_1 is bounded from below (resp. above) for $\lambda < 1$ (resp. $\lambda > 1$). We stick with the case $\lambda < 1$ for concreteness. Suppose $M_1 \eta = \mu \eta$ with $\mu = inf spec M_1 \leq 0$. Applying the maximum principle (eg proposition 6 for $d = 1$) to (41), we conclude that $\eta > 0$. If $\mu < 0$, we have $< \eta, f' >= 0$, a contradiction. Thus $\mu = 0$, and is non-degenerate by a similar argument. $\square$

We also have

Lemma 1 *For $m \geq 2$, $M_m - M_1$ is non-negative for $\lambda < 1$, non-positive for $\lambda > 1$, and has no zero-eigenvalue.*

Proof: This follows from the equation

$$M_m - M_1 = (1 - q^2)\frac{m^2 - 1}{r^2}. \qquad \square$$

Completion of the proof of theorem 1: Suppose now $\lambda < 1$. Since $\tilde{F}_m^* \tilde{F}_m$ is manifestly non-negative, and $M_m > M_1$ for $m \geq 2$, we have $L_m^{(n)} \geq 0$ for $m \geq 1$ (with only the translational 0-mode). Combined with (26) and propositions 7 and 3, this gives stability of the n-vortex for $\lambda < 1$.

Now suppose $\lambda > 1$. By (40), proposition 10 and lemma 1, we have for $m = \pm 2, \ldots \pm n$,

$$< \tilde{W}_m, L_m^{(n)} \tilde{W}_m > \; < \; 0.$$

We remark that $\tilde{W}_m$ corresponds to an element of the un-complexified space X, and so $L^{(n)}$ has negative eigenvalues. This establishes the instability of the n-vortex for $|n| \geq 2$, $\lambda > 1$, and completes the proof of theorem 1. $\square$

8 Appendix: vortex solutions are radial minimizers

Proposition 11 *For $\lambda \geq 2n^2$, a solution of the equations (11-12) minimizes $E_r^{(n)}$.*

Proof: It suffices then to show $M_0 = HessE_r^{(n)} > 0$ (see section 5.1). We write $M_0 = L_0 + Z_0$ where

$$L_0 = diag\{l, -\Delta_r\}$$

with $l = -\Delta_r + b^2 + \frac{\lambda}{2}(f^2 - 1)$ and

$$Z_0 = \begin{pmatrix} 2\lambda f^2 & -2bf \\ -2bf & \frac{1}{r^2} + f^2 \end{pmatrix}.$$

We note that $lf = 0$ (one of the GL equations). It follows from the fact that $f > 0$ and a Perron-Frobenius type argument (see [OS1]) that $l \geq 0$ with no zero-eigenvalue. It suffices to show $Z_0 \geq 0$. Clearly $tr(Z_0) > 0$, and

$$\det(Z_0) = 2\lambda f^4 + \frac{2f^2}{r^2}[\lambda - 2n^2(1-a)^2]$$

is strictly positive for $\lambda \geq 2n^2$. □

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